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Painlevé analysis of the generalized Burgers–Huxley equation

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Abstract. In this paper a complete Painlevé test is applied to the generalized Burgers-Huxley equation using the version of the Painlevé analysis recently developed by Weiss, Tabor and Carnevale for partial nonlinear differential equations. In so doing, we are able to find a complete set of new solutions as well as recovering some previous particular solutions already found by using *ad hoc* methods which have been recently published.

1. Introduction

Almost any branch of physics is nowadays unavoidably associated with problems involving nonlinear differential equations which logically need to be integrated. These differential equations can be either partial differential equations (PDE) or ordinary differential equations (ODE), the latter sometimes arising as a special reduction of the former. In either of these cases the available tools one has for determining whether these differential equations are integrable or not are group theory and Painlevé tests. In the first case one looks for groups leaving the equations invariant and this avenue leads to the study of infinite-dimensional Lie algebras, Kac-Moody algebras and the like. In the second case one is led to analyse the analytical behaviour in the complex plane of the singularities of the solution manifold giving rise to important achievements such as Bäcklund transformations, integrability tests and finally explicit solutions.

In this paper we fully apply the heavy artillery of the Painlevé analysis [1-5] to the generalized Burgers-Huxley equation. We do this for two reasons. First because this equation has several interesting limiting cases which have already been studied. We will be talking about these limits below. The second reason is that a recent paper has appeared [6] in which a very limited set of solutions of this equation is presented without any systematical analysis. We shall show that using the full-power Painlevé analysis those solutions appear as particular cases of the very general solutions here presented. This shows again that in dealing with solutions either of PDE or ODE, one needs to use a systematic method providing a full insight into the corresponding nonlinear problem since particular clever tricks only increase the number of papers but not the desirable knowledge of the physical system associated with such nonlinear evolution equations.

The generalized Burgers-Huxley equation is a diffusion equation which has the form [6]

$$\omega_t + \alpha\omega^m\omega_x - \omega_{xx} = \beta\omega(1-\omega^m)(\omega^m-\gamma) \tag{1}$$

where α, β, γ and m are constant parameters. The case $m = 1$ contains several known evolution equations:

(a) when $\beta = 0$ is the Burgers equation [5];

(b) when $\alpha = 0$ is the Huxley equation, sometimes known as the FitzHugh–Nagumo equation [7];

(c) when $\alpha = 0$ and $\gamma = -1$ is a particular case of the above, known as the Newell–Whitehead equation [8].

Our aim in this paper is to apply the Painlevé analysis to equation (1). However, we have to use the modified version of the Painlevé analysis developed and thoroughly used by Weiss *et al* [5] and suitably adapted to the PDE case. The reader is also referred to [9, 10] for a complete set of references on the subject.

Even in those cases in which no integrability is found we can find particular solutions by imposing truncation in the Painlevé series according to the procedure developed by Cariello and Tabor [8]. We shall find in this way solutions of (1) that will contain as particular cases those found in [6–8].

The plan of this paper is the following. In section 2 we revise the necessary basics of the Weiss–Tabor–Carnevale method: Painlevé analysis applied to PDE. We also apply it to equation (1) and discuss its integrability. In section 3 we use the truncation method for finding particular solutions which we also classify according to a systematic method. A comparison is made with other results obtained from our more general solutions as particular cases. We close with some conclusions and comments on the solitonic nature of our solutions.

2. The Weiss–Tabor–Carnevale method applied to the generalized Burgers–Huxley equation

Let us write equation (1) in the form

$$uu_t - uu_{xx} + \frac{m-1}{m} u_x^2 + \alpha u^2 u_x + \beta m u^2 (u-1)(u-\gamma) = 0 \tag{2}$$

which can be obtained from (1) through the obvious transformation $u = w^n$. In order to obviate bad singularities at $u = 0$ (the coefficient of u_{xx}) we will be restricted to the cases with m integer [11]. A given PDE is said to have the Painlevé property when its solutions are single valued about the movable singularity manifolds. A movable singularity is a singularity which depends upon the initial conditions. The Painlevé test as it was formulated in [2] established that whenever a PDE is solvable through the inverse scattering method, the associated ODE obtained as a similarity reduction of the PDE [10] is of Painlevé type: the only movable singularities are poles.

Later on, a method specially designed for PDE without relying on the associated ODE (nor the inverse scattering method) was put forward by Weiss *et al* [5]. It is this method which will be applied to our equation (2). To begin with let us express the solutions of (2) in the form

$$u(x, t) = \phi^r \sum_{j=0}^{\infty} u_j \phi^j \tag{3}$$

where $\phi(x, t) = 0$ is the singularity manifold of the equation while the u_j are functions of x and t, analytic about a neighbour of this manifold.

The leading index r as well as the recurrence relations among the u_j are easily obtained by inserting (3) into (2). The result is that the leading index is $r=-1$ and the recurrence relations are

$$\begin{aligned}\sum_{n=0}^{j} [u_n(u_{j-n-2})_t &+ (n-1)\phi_t u_n u_{j-n-1} - u_{j-n-2}(u_n)_{xx} - 2(n-1)\phi_x u_{j-n-1}(u_n)_x \\ &- (n-1)\phi_{xx} u_n u_{j-n-1} - (n-1)(n-2)\phi_x^2 u_n u_{j-n}] \\ &+ \frac{m-1}{m} \sum_{n=0}^{j} [(u_n)_x (u_{j-n-2})_x + 2(n-1)\phi_x u_n (u_{j-n-1})_x \\ &+ (n-1)(j-n-1)\phi_x^2 u_n u_{j-n}] \\ &+ \alpha \sum_{k=0}^{j} \sum_{n=0}^{j} [(u_n)_x u_k u_{j-n-k-1} + (n-1)\phi_x u_n u_k u_{j-n-k}] \\ &+ \beta m \sum_{k=0}^{j} \sum_{p=0}^{j} \sum_{n=0}^{j} u_n u_k u_p u_{j-n-p-k} \\ &- \beta m(\gamma+1) \sum_{k=0}^{j} \sum_{n=0}^{j} u_n u_k u_{j-n-k-1} + \beta m \gamma \sum_{n=0}^{j} u_n u_{j-n-2} = 0. \end{aligned} \tag{4}$$

In particular, for $j=0$,

$$u_0 = -\lambda \phi_x \tag{5}$$

where λ is a constant verifying

$$\beta m^2 \lambda^2 + \alpha m \lambda - (m+1) = 0. \tag{6}$$

The u_j coefficient for $j>0$ in (4) is

$$\lambda \phi_x^3 (j+1)\left[j - \left(2\frac{(1+m)}{m} - \alpha\lambda\right)\right]. \tag{7}$$

Therefore, the equation presents a resonance when the following relation holds:

$$\frac{2(1+m)}{m} - \alpha\lambda = k \tag{8}$$

where k is a positive integer. In this case u_k is an arbitrary function and for the equation to be integrable one needs that the correspondent $j=k$ term in (4) verifies the relationship identically. We have checked by substitution in (4) that for $k=1$ in (8) the resonance condition is verified and the equation is totally integrable. For $k=2, 3$ the resonance condition is not verified and the equation is not totally integrable. In particular the case $k=1$ will be thoroughly analysed in the last part of the next section in which the truncation method will be applied.

3. Truncation method and particular solutions

Recently it was observed by Cariello and Tabor [8] that a procedure for obtaining particular solutions of a non-integrable PDE can always be found by looking for the particular solutions of the equation such that the resonance condition is verified by imposing the necessary conditions for the truncation of the expansion. In our case, we note that the only way to obtain a finite recursion expansion in (4) is $u_1 = u_2 = 0$.

In this case all terms with $j > 2$ of the expansion vanish and the resonance condition, if it exists, will also be satisfied since all terms appearing in it are zero. If such a condition is actually imposed we are led to a function u satisfying:

$$u = -\lambda(\phi_x/\phi) \tag{9a}$$

$$p = (\phi_t/\phi_x) \tag{9b}$$

$$q = (\phi_{xx}/\phi_x) \tag{9c}$$

where p and q verify the equations

$$p = \left(\frac{m+2}{m} - \alpha\lambda\right) q + \beta m\lambda(\gamma+1) \tag{10a}$$

$$q_x = -\frac{\beta m^2\lambda^2}{2-\alpha\lambda m}\left(q+\frac{1}{\lambda}\right)\left(q+\frac{\gamma}{\lambda}\right). \tag{10b}$$

The equation (10*b*) has three different types of solutions which in turn will result in three different types of solutions for ϕ through the integration of (9*a*, *b*).

Case 1.

$$q = -\frac{1}{\lambda}. \tag{11}$$

In this case $\phi = A(t) - \lambda \exp\{-(1/\lambda)(x+g(t))\}$ and imposing (9*b*) we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 0 \tag{12}$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \beta m\gamma\lambda - (m\lambda)^{-1} \tag{13}$$

and then we obtain for $u(x, t)$

$$u(x, t) = \left[1 + B\exp\left\{\frac{1}{\lambda}(x-ct)\right\}\right]^{-1} \tag{14a}$$

$$c = (m\lambda)^{-1} - \beta m\gamma\lambda \tag{14b}$$

where B is an arbitrary constant.

Case 2.

$$q = -\frac{\gamma}{\lambda}. \tag{15}$$

In a similar manner we obtain for ϕ

$$\phi(x, t) = A(t) - \left(\frac{\lambda}{\gamma}\right)\exp\left\{-\frac{\gamma}{\lambda}(x+g(t))\right\} \tag{16}$$

and imposing (9*b*) we find

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 0 \tag{17}$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \beta m\lambda - \frac{\gamma}{m\lambda} \tag{18}$$

and then we obtain for $u(x, t)$

$$u(x,t)=\gamma\left[1+B\exp\left\{\frac{\gamma}{\lambda}(x-ct)\right\}\right]^{-1} \tag{19a}$$

$$c=\frac{\gamma}{m\lambda}-\beta m\lambda \tag{19b}$$

again B is an arbitrary constant.

The solutions (14a, b) and (19a, b) are the travelling-wave solutions obtained in [6]. Also $\alpha=0$ and $m=1$ correspond to two of the solutions of the FitzHugh–Nagumo equation described in [7].

Case 3.

$$q=-\frac{1}{\lambda}\left(\frac{1-\gamma\exp\{a(1-\gamma)(x+g(t))\}}{1-\exp\{a(1-\gamma)(x+g(t))\}}\right) \tag{20}$$

where

$$a=(\beta m^2\lambda)(2-\alpha m\lambda)^{-1}. \tag{21}$$

For the general conditions (9a–c) and (10a, b) we obtain

$$\phi_x=\exp\{h(t)\}[\exp\{-\gamma a(x+g(t))\}-\exp\{-a(x+g))\}]^{1/\lambda a} \tag{22a}$$

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t}=-\frac{(1+\gamma)}{(m\lambda)} \tag{22b}$$

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t}=\left(\frac{\alpha\gamma}{\lambda^2}\right)\left(\lambda-\frac{m+2}{m\alpha}\right) \tag{22c}$$

$$(m-1)\left(\lambda-\frac{m+2}{m\alpha}\right)=0. \tag{22d}$$

From the last condition we see that solutions would only exist if $m=1$ or $\lambda=(m+2)/(m\alpha)$. We shall investigate these two cases separately.

Case 3a. Using (22a–c) with $m=1$ and (9a–c) one easily finds:

$$u(x,t)=\gamma\left(\frac{\exp\{-(\gamma/\lambda)(x-ct+x_0)\}-\exp\{-(1/\lambda)(x-ct+x_0)\}}{B\exp\{(\gamma/\lambda^2)(2-\lambda\alpha)t\}+\exp\{-(\gamma/\lambda)(x-ct+x_0)\}-\gamma\exp\{-(1/\lambda)(x-ct+x_0)\}}\right) \tag{23a}$$

and

$$c=\frac{(1+\gamma)}{\lambda}. \tag{23b}$$

The limits $x_0\to\infty$, $x_0\to-\infty$ and $B\to 0$ for $\alpha=0$ yield the three travelling-wave solutions of the FitzHugh–Nagumo equation described in [7] with different wave velocities to those mentioned in cases 1 and 2. Also the case $\alpha=0$ and $\gamma=-1$ corresponds to the Newell–Whitehead equation whose solutions have been extensively described by Cariello and Tabor [8]. In these limits (23) trivially yields:

$$u(x,t)=\frac{\sinh[2^{-1/2}(x+x_0)]}{\cosh[2^{-1/2}(x+x_0)]+2B\,\mathrm{e}^{-3t/2}} \tag{24}$$

which is indeed the same as the one obtained in [8].

Case 3b. Let us now take $\lambda = (m+2)/(m\alpha)$. From equations (6) and (21) we find the following relation among the parameters:

$$\beta(m+2)^2 = -\alpha^2 \qquad \text{and} \qquad a\lambda = \frac{1}{m}. \tag{25}$$

We would like to point out at this stage that in this case the expansion (4) shows a resonance in $j=1$ and in addition the resonance condition is identically satisfied. Thus, when the parameters of equation (2) satisfy (25) we have a totally integrable equation with a resonance in $j=1$. Therefore the function u_1 is arbitrary and when we choose $u_1=0$ the obtained solution will correspond to the most general solution obtained as a truncated expansion. Using (22*a*–*c*) and (9*a*–*c*) together with (25) we obtain after lengthy calculation the solution for $u(x, t)$:

$$u(x, t) = \frac{S_1(x-ct+x_0)}{S_2(x-ct+x_0)} \tag{26}$$

where, if $z = x-ct+x_0$, $S_1(z)$ and $S_2(z)$ can be written as:

$$S_1(z) = \left[\exp\left\{-\frac{\alpha\gamma}{m+2}z\right\} + \exp\left\{-\frac{\alpha}{m+2}z\right\}\right]^m \tag{27a}$$

$$S_2(z) = B + \sum_{k=0}^{m} \frac{m}{m-k(1-\gamma)}\binom{m}{k}\exp\left\{-\frac{\alpha(m-k(1-\gamma))}{m+2}z\right\} \tag{27b}$$

and $c = \alpha(1+\gamma)/(m+2)$. Indeed B in an arbitrary constant.

4. Conclusions and comments

Aside from the solutions obtained in cases 1 and 2, which have already been found by other authors, case 3 presents a totally new set of solutions. Special cases of case 3*a* have been the subject of previous contributions. The solution in case 3*b* is totally new. At any rate, it is important to point out that all these solutions represent a sort of generalized asymmetric kink. More precisely, solution (26) and (27*a*, *b*) behaves at $x \to \pm\infty$ as

$$u(x, t) \underset{x\to-\infty}{\to} \frac{1}{2}\left[1 - \tanh\left(\frac{\alpha m}{2(m+2)}z\right)\right] \tag{28a}$$

$$u(x, t) \underset{x\to+\infty}{\to} \frac{\gamma}{2}\left[1 - \tanh\left(\frac{\alpha\gamma m}{2(m+2)}z\right)\right] \tag{28b}$$

which are the solutions (14*a*, *b*) and (19*a*, *b*). Then, our solution (26) and (27*a*, *b*) interpolates between previous known solutions much as it happens in other cases [8]. Whether or not our solution represents a multikink solution remains to be seen by performing numerical asymptotic calculations which will be reported elsewhere.

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